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# DIGITAL SIMULATION OF CONTINUOUS ERROR MODELS WITH APPLICATION TO AN INSTRUMENT LANDING SYSTEM ERROR

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## ABSTRACT

During the course of a study of space shuttle navigation capabilities and requirements, it became necessary to construct a digital simulation of the continuous error of the localizer beam of a conventional instrument landing system. A discrete mathematical model has been developed which is easy to use on a digital computer. This model is a system of difference equations driven by a zero-mean gaussian random sequence. The model generates an output random sequence which is equivalent, for simulation purposes, to the desired random process. The equivalence is created by requiring that the first and second order statistics of the discrete model be equal to the corresponding expectations of the random process to be simulated at all instants of time which occur in the simulation.

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## SUMMARY

During the course of a study of space shuttle navigation capabilities and requirements, it became necessary to construct a digital simulation of the continuous error of the localizer beam of a conventional instrument landing system. A discrete mathematical model has been developed which is easy to use on a digital computer. This model is a system of difference equations driven by a zero-mean gaussian random sequence. The model generates an output random sequence which is equivalent, for simulation purposes, to the desired random process. The equivalence is created by requiring that the first and second order statistics of the discrete model be equal to the corresponding expectations of the random process to be simulated at all instants of time which occur in the simulation.

The theory is presented in several distinct sections. Then the details of a localizer beam numerical example are given using the same format.

## INTRODUCTION

While participating in a study of space shuttle navigation capabilities and requirements, it became necessary to construct a mathematical model for the digital simulation of the continuous random error of the localizer beam of a conventional instrument landing system. The localizer beam error random process is nonstationary since the rms error decreases with range; however, it has been assumed (ref. 6) that the frequency distribution characteristics of this error are described by a rational power spectral density function which does not change with time. Although a method is well-known (e.g., ref. 1) for obtaining a continuous-time representation of a stationary markovian random process from a given rational power spectral density function, it is not so obvious how to obtain a suitable discrete-time representation for easy digital computer implementation. The purpose of this paper is to present a straightforward computationally efficient method for constructing such a model and show that it may be used for the practical problem at hand. The general approach is one suggested by Dr. Stanley F. Schmidt.

# NOTATION

## COLUMN VECTORS (lower case, underlined)

| | |
|---|---|
| $\underline{x}_c$ | an n component differential system response, the state vector |
| $\underline{x}_d$ | a random sequence which is statistically equivalent to $\underline{x}_c$ at the discrete time points $t_k$ |
| $\underline{y}$ | m elements, a linear function of x, $\underline{y} = H\underline{x}$ |
| $\underline{u}$ | r elements--each element is a white noise random sequence |
| $\underline{w}$ | n elements--each element is a white noise process |

## MATRICES (capital leters)

| | |
|---|---|
| B | $n \times n$, symmetric |
| F | $n \times n$, a known function of time |
| G | $n \times n$, a known function of time |
| H | $m \times n$, a known function of time |
| $P_{x_c}$ | the expected value of the matrix, $\underline{x}_c \underline{x}_c^T$, the covariance matrix of $\underline{x}_c$ |
| $P_{x_d}$ | the expected value of the matrix, $\underline{x}_d \underline{x}_d^T$ |
| R | the expected value of the matrix, $\underline{w}\,\underline{w}^T$ |
| $\Phi(t:to)$ | the matrix which transitions the state vector from one time to another |
| $\gamma$ | $n \times r$, to be determined |
| $(\ )^T$ | the transpose of the matrix ( ) |

## GENERAL

| | |
|---|---|
| a | a constant |
| s | the laplace operator |

| | |
|---|---|
| $t$ | time |
| $\tau$ | a particular time |
| $\Delta T$ | time interval, $t - t_o$ or $t_k - t_{k-1}$ |

SUBSCRIPTS

| | |
|---|---|
| c | continuous |
| d | discrete |
| k | an integer |
| o | initial |
| s | steady state |

# THEORY

## 1. Preliminary Considerations

The motivation for this theoretical development was the need for a digital computer algorithm to model the continuous error of an instrument landing system localizer beam. It was desired that the algorithm be capable of generating a random sequence statistically equivalent to the random process to be simulated, and that the algorithm be reasonably simple.

Unless the given random process happens to be gauss-markov, or some other type completely characterized by only a few parameters, the objectives stated above tend to be mutually contradictory. Therefore, a compromise was employed which is common in engineering applications; namely, that only first- and second-order statistical equivalence would be required. Specifically, at each time point for which the random sequence is defined, its first- and second-order statistics were required to be equal to those of the random process being simulated:

$$E\{\underline{x}_d(t_k)\} = E\{\underline{x}_c(t_k)\}, \quad E\{\underline{x}_d(t_k)\underline{x}_d^T(t_1)\} = E\{\underline{x}_c(t_k)\underline{x}_c^T(t_1)\}$$

The justifications for this compromise are:

(1) The description of a physical random process if often limited to approximating its first- and second-order statistics (such is the case with the example used here).

(2) First- and second-order statistics are commonly the most significant probabilistic attributes of the process.

(3) The well-developed theory of gauss-markov processes and sequences can be applied readily.

These principles and assumptions are applied in the sequel to develop the desired algorithm, the steps in the development being summarized as follows:

(a) From a given rational power spectral density function which represents (in the frequency domain) the second-order statistics of the error random process, develop a differential equation with zero-mean white noise input whose solution would produce elements of the desired random process.

(b) Develop from this differential equation a general relationship for the steady-state covariance matrix of the state of this continuous system, expressed in terms of the coefficients of the differential equation.

(c) Develop a general expression for the steady-state covariance matrix of the state of a discrete system with a zero-mean white noise random sequence input in terms of the system coefficients.

(d) Equate the covariance expressions for the continuous and discrete systems and solve for the coefficients of the required discrete system.

The discrete system (i.e., digital computer algorithm) designed by this procedure produces a zero-mean random sequence having the desired second-order statistical properties. Usually, as in the example considered herein, the process to be simulated is a zero-mean process, or by proper problem definition can be made so. However, if not, the desired first-order statistical properties can be achieved simply by adding to the generated random sequence, a sequence which is the mean of the given error random process at the defined time points.

A certain degree of added generality is obtained automatically in the development inasmuch as generally an n-state system must be devised in order to model a random process of anything more than rudimentary complexity. Since the n-state system has $n$ outputs, and since an infinite variety of combinations of these outputs is possible, one can model not only the desired random process, but at the same time many other related random processes with little additional computation. In particular, it will be noted that a variety of non-stationary random processes can be modeled simply by multiplying the system output by an arbitrary time-varying factor. That is, if $\underline{x}_d$ is the n-component state vector of the system, and $H$ is an $(m \times n)$ matrix of time-varying coefficients, then

$$\underline{y}_d = H\underline{x}_d$$

is a vector random sequence whose $m$ components may have arbitrary variances at any given time.

2. The Covariance Matrix for a Continuous Error Model (a gauss-markov random process)

A state vector formulation of a linear differential system response is

$$\dot{\underline{x}}_c = F(t)\underline{x}_c + G(t)\underline{w} \tag{2}$$

where $\underline{x}_c$ is an $n$ element state vector, $w$ is an $n$ element noise vector, and the $n \times n$ matrices, F and G, are known functions of time. In order that we may apply the useful theory that exists for gauss-markov random processes, we now require:

a) $\underline{w}$ is zero-mean gaussian white noise

b) $\underline{x}_c(o)$ has a zero-mean gaussian distribution

c) $E[\underline{w}\ \underline{x}_c^T(o)] = 0$

Then $\underline{x}_c$ is a representation of a zero-mean gauss-markov random process and

$$E\{\underline{w}(t)\underline{w}^T(\tau)\} = R\delta(t - \tau)$$

where R is a positive semi-definite $n \times n$ matrix and $\delta$ is the Dirac delta function.

For this application, we also require that $\underline{w}$ be stationary so that R is a constant. The associated covariance propagation equation is (ref. 2 and 3)

$$\dot{P}_{x_c}(t) - FP_{x_c} + P_{x_c}F^T + GRG^T \tag{3}$$

The solution of the state equation is

$$\underline{x}_c(t) = \Phi(t:t_0)\underline{x}_c(t_0) + \int_{t_0}^{t} \Phi(t:u)G(u)\underline{w}(u)du \tag{4}$$

where $\Phi(t:t_0)$ is the transition matrix (refs. 4, 5) and the expected value of the integral is zero. Equation (4) is a continuous representation for the state vector at any time $t$ in terms of initial conditions at any time $t_0$, the transition matrix, and the disturbing functions. The

transition matrix for a linear constant coefficient differential system ($F$ = constant) is a function only of the time difference, $t - t_0$; it is not a function of time, $t$.

It is readily determined that

$$P_{x_c}(t) = \Phi(t:t_0)P_{x_c}(t_0)\Phi^T(t:t_0) + B \tag{5}$$

where $B$ is a symmetric $n \times n$ matrix and

$$B = E\{\int_{t_0}^{t} \Phi(t:u)G(u)\underline{w}(u)du(\int_{t_0}^{t} \Phi(t:v)G(v)\underline{w}(v)dv)^T\} \tag{6}$$

It is ordinarily also true (ref. 3) that

$$B = \int_{t_0}^{t} \Phi(t:v)G(v)RG^T(v)\Phi^T(t:v)dv \tag{7}$$

Here the integrand is positive semi-definite. Consequently, the integral $B$ is also positive semi-definite.

In any application, the covariance matrix of the continuous random process must be explicitly determined. Accordingly, an expression for the integral $B$ is needed. The integration is tedious and may be avoided if, as in most applications, the random process to be modeled is independent of time. When this is true the stable system, which generates the process, is in a steady-state condition; the matrices $GRG^T$ and $F$ are constant, and the transition matrix is a function of the time interval $\Delta t$. In the steady-state condition, equations (3) and (5) become:

$$0 = FP_{x_c}(t_s) + P_{x_c}(t_s)F^T + GRG^T \tag{8}$$

$$P_{x_c}(t_s) = \Phi(\Delta t)P_{x_c}(t_s)\Phi^T(\Delta t) + B \tag{9}$$

where the notation $t_s$ means any time such that the system is in a steady-state condition.

We see that equation (8) determines the elements of the steady-state continuous covariance matrix in terms of known constants and equation (9) defines $B$ in terms of the $P$ matrix and the transition matrix. Thus, $B$ has been determined without integration.

Continuing the general development and using equations (1), (2), and (5), we write:

$$E(\underline{y}_c) = 0 \tag{10}$$

$$P_{y_c}(t) = H(t)P_{x_c}(t)H^T(t) \tag{11}$$

$$P_{y_c}(t) = H(t)\Phi(t:t_o)P_{x_c}(t_o)\Phi^T H^T + HBH^T \tag{12}$$

where $P_{y_c}(t)$ is defined by

$$P_{y_c}(t) \triangleq E[\underline{y}_c(t)\underline{y}_c^T(t)] \tag{13}$$

Equations (5), (7), and (12), present the statistical description of the random process which is to be simulated with difference equations and discrete random variables.

3. <u>The Covariance Matrix for a Discrete Error Model (a gauss-markov random sequence)</u>

We wish to generate an $\underline{x}_d(t_k)$ such that $P_{x_d}(t_k) = P_{x_c}(t)$ at all times $t_k$, occurring in the digital simulation. We start with the continuous equation (4) which has the following discrete analog

$$\underline{x}_d(t_k) = \Phi(t_k:t_{k-1})\underline{x}_d(t_{k-1}) + \gamma(t_k)\underline{u}(t_k) \tag{14}$$

where $\underline{x}_d$ is an $n$ component vector whose elements are random sequences with an initial gaussian distribution. Here $\underline{u}$ is an $r$ component vector of gaussian-random sequences with the following characteristics:

$$E[u_i(t_k)] = 0 = E\,[u_i(t_k)\underline{x}_d(t_{k-1})]$$

$$E[u_i^2(t_k)] = 1 \text{ for } i = 1 \text{ to } r,\ r \geq 1$$

$$E[u_i(t_k)u_j(t_k)] = 0 \text{ if } i \neq j$$

Consequently,

$$E[\underline{u}(t_k)\underline{u}^T(t_k)] = I \tag{15}$$

The vector $\underline{u}(t_k)$ is easily obtained from a unit variance gaussian-random number generator.

Now $\underline{x}_d$ is a representation of a gauss-markov sequence and

$$P_{x_d}(t_k) = \Phi P_{x_d}(t_{k-1})\Phi^T + \gamma(t_k)\gamma^T(t_k) \tag{16}$$

which is the discrete covariance relationship corresponding to the continuous relationship expressed in equation (5).

If we wish to model a random process, which is a linear function of a differential equation response as suggested in equation (1), we introduce:

$$\underline{y}_d(t) = H(t)\underline{x}_d(t) \tag{17}$$

$$P_{y_d}(t) = H(t)P_{x_d}(t)H^T(t) \tag{18}$$

and find that

$$P_{y_d}(t) = H\Phi P_{x_d}(t_o)\Phi^T H^T + H\gamma\gamma^T H^T \tag{19}$$

4. Requirements for the Equivalence of the Continuous and Discrete Covariance Matrices

The two equations which define the manner in which the covariance matrices propagate, are repeated here:

$$P_{x_c}(t) = \Phi(t:t_o)P_{x_c}(t_o)\Phi^T(t:t_o) + B \tag{5}$$

$$P_{x_d}(t_k) = \Phi(t_k:t_{k-1})P_{x_d}(t_{k-1})\Phi^T(t_k:t_{k-1}) + \gamma(t_k)\gamma^T(t_k) \tag{16}$$

An inspection of these relationships shows that the discrete variable will be statistically equivalent to the continuous quantity if the following three conditions are met:

(a) $\Phi(t_k:t_{k-1}) \equiv \Phi(t:t_o)$

(b) The discrete variable can be initialized such that

$$P_{x_d}(t_{k-1}) = P_{x_c}(t_o)$$

(c) A matrix $\gamma$ can be found such that $\gamma\gamma^T = B$ (20)

The example to be presented will illustrate, in detail, how these conditions are satisfied in a particular case.

The other covariance relationships of equation (19) and equation (12) are also seen to be statistically equivalent if the preceding three conditions are satisified.

## EXAMPLE

### 1. Preliminary Discussion

A signal whose power spectral density approximates that of the localizer noise (ref. 6) can be generated by passing gaussian white noise through a filter as indicated below:

$$\xrightarrow[\text{noise}]{\text{white}} \boxed{\dfrac{S}{1 + 3.0S + 2.25S^2}} \longrightarrow \boxed{k} \xrightarrow[\text{noise}]{y\ =\ \text{localizer}}$$

It is assumed that initial transients have disappeared so that the statistical expectation of the output of the transfer function is not time dependent. The gain $k$ is adjusted as a function of range so that the variance of the localizer noise representation, $y$, is appropriate.

When white noise is passed through a filter of the form

$$\frac{1}{1 + (2/a)S + (1/a)^2S^2}$$

then when $a = 2/3$ the derivative of the output has the statistical distribution we desire though its magnitude must be adjusted.

### 2. An Explicit Presentation of the B Matrix

A state vector form of the differential equations which represent the localizer system is

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a^2 & -2a \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ w_2 \end{bmatrix} \tag{21}$$

$$y = \begin{bmatrix} 0 & k \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{22}$$

where $w_2$ is a gaussian white noise process of zero-mean and unit variance and the system of equations is analogous to equations (1) and (2). In this example the vector of noise sources, $\underline{w}$, is characterized by

$$E[\underline{w}(t)w^T(\tau)] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \delta(t - \tau) \tag{23}$$

so that

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

If we define

$$P_{x_c}(t_s) \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

$$\Phi(\Delta T) \triangleq \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$B \triangleq \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

then equation (8) may be applied to this example to get

$$\begin{aligned} P_{12} &= 0 \\ P_{22} &= a^3/4 \\ P_{11} &= P_{22}/a^2 \end{aligned} \tag{24}$$

Next, applying equation (9) we have

$$\begin{aligned} b_{11} &= (a/4)[1 - \phi_{11}^2 - \phi_{12}^2 a^2] \\ b_{12} &= (a/4)[-\phi_{11}\phi_{21} - \phi_{22}\phi_{12}a^2] \\ b_{22} &= (a/4)[a^2 - \phi_{21}^2 - \phi_{22}^2 a^2] \end{aligned} \tag{25}$$

In Appendix A, the transition matrix for this application is determined to be

$$\phi_{11} = (1 + a\Delta t)e^{-a\Delta t}$$

$$\phi_{12} = \Delta t e^{-a\Delta t}$$

$$\phi_{21} = -a^2\Delta t e^{-a\Delta t}$$

$$\phi_{22} = (1 - a\Delta t)e^{-a\Delta t} \tag{26}$$

and we may find

$$b_{11} = (a/4) - (a/4)e^{-2a\Delta t}[1 + 2a\Delta t + 2a^2(\Delta t)^2] \tag{27}$$

$$b_{12} = (a/4)e^{-2a\Delta t}[+2a^3(\Delta t)^2] \tag{28}$$

$$b_{22} = (a/4)a^2 - (a/4)a^2e^{-2a\Delta t}[1 - 2a\Delta t + 2a^2(\Delta t)^2] \tag{29}$$

3. Determination and Existence of $\gamma$

The components of B have been explicity presented; for a given a and $\Delta t$ the numerical values of the elements of B can be calculated. The next step in obtaining a solution to equation (14) is to determine a $\gamma$ such that

$$\gamma\gamma^T = B \tag{17}$$

A lower triangular form for $\gamma$ is sufficient here

$$\gamma \triangleq \begin{bmatrix} \gamma_{11} & 0 \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$

$$\gamma\gamma^T = \begin{bmatrix} \gamma_{11}^2 & \gamma_{11}\gamma_{21} \\ \gamma_{11}\gamma_{21} & \gamma_{21}^2 + \gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \tag{30}$$

We now have three algebraic equations in three unknowns which are readily solvable. If the matrices, $\gamma$ and B, had n rows and columns, then there would be $n(n + 1)/2$ equations in the same number of unknowns to solve but their solution is straightforward. Appendix B presents the details for matrices with three rows and columns.

For the present example:

$$\gamma_{11} = b_{11}^{1/2} \tag{31}$$

$$\gamma_{21} = b_{12}/\gamma_{11} \tag{32}$$

$$\gamma_{22} = (b_{22} - b_{12}^2/b_{11})^{1/2} \tag{33}$$

It is clear that the elements of $\gamma$ do exist if

$$b_{11} > 0 \text{ and } b_{11}b_{22} - b_{12}^2 \geq 0$$

The matrix $B$ is positive semi-definite, which implies that:

$$b_{11} \geq 0, \; b_{11}b_{22} - b_{12}^2 \geq 0$$

If $b_{11} = o$ it follows that $b_{12}$ is also equal to zero and in this case

$$\gamma = \begin{bmatrix} 0 & 0 \\ 0 & b_{22}^{1/2} \end{bmatrix}$$

When the matrix $B$ is less than full rank, then the vector $\underline{u}$ may be constructed with fewer elements.

## 4. Initialization of the Discrete Variable

All of the major elements necessary for solving equation (14) have now been discussed. The random vector $\underline{u}$ is easily obtainable, a transition matrix can be calculated, and the matrix $\gamma$ has been presented for the example problem.

The remaining requirement is to determine an initializing vector $\underline{x}_d(o)$. This is simple in the given example since the cross correlation term of the covariance matrix, $P_{12}$, is zero. We wish to construct initial values $x_d(o)$ and $\dot{x}_d(o)$ such that

$$E[x_d^2(o)] = P_{11} = a/4$$

$$E[\dot{x}_d^2(o)] = P_{22} = a^3/4$$

$$E[x_d(o)\dot{x}_d(o)] = 0$$

Take an output from a unit variance gaussian zero-mean random number generator (uncorrelated) and multiply this number by $P_{11}^{1/2}$; this is $x_d(o)$. Take a different element from this same gaussian sequence and multiply it by $P_{22}^{1/2}$; this is $\dot{x}_d(o)$. These elements form the initial state vector.

## CONCLUDING REMARKS

In summary, the example random process is modeled for computation by programming the following equations

$$\underline{x}_d(t_k) = \Phi(\Delta t)\underline{x}_d(t_{k-1}) + \gamma(t_k)\underline{u}(t_k) \tag{34}$$

$$y(t_k) = [0 \quad K]\underline{x}_d(t_k) \tag{35}$$

where $\Phi(\Delta t)$ is given by equation (26), and $\gamma(t_k)$ is obtained from equations (27) through (29) and (31) through (33). Initial conditions for equations (34) and (35) are

$$\underline{x}_d(o) = (a/4)^{1/2}\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}\underline{u}(o)$$

and $\underline{u}(t_k)$ is a 2-component vector whose elements are obtained from a gaussian unit-variance random number generator.

The problem of simulating a statistically defined continuous error on a digital computer has been discussed. A technique for creating difference equations which fulfil this purpose has been presented. This mathematical model is easy to mechanize and it is accurate in the sense that the vector of random sequences obtained from it is statistically equivalent, at the discrete times, $t_k$, to the vector of random processes being modeled. A digital simulation has been constructed for the random process which describes the random error of a conventional instrument landing system localizer beam.

# APPENDIX A

## A DERIVATION OF THE TRANSITION MATRIX FOR THE EXAMPLE APPLICATION

$$N(t) \longrightarrow \boxed{\frac{1}{1 + (2/a)S + (1/a^2)S^2}} \longrightarrow x(t)$$

The corresponding equation is

$$(S^2 + 2aS + a^2)L\{x(t)\} = a^2 L\{N(t)\} \tag{A1}$$

When $N(t)$ is identically zero in the equation above, the solution vector $\underline{x}(t_n)$ is related to the same vector at an earlier time, $\underline{x}(t_{n-1})$, by a *multiplying matrix as below*

$$x(t_n) = \Phi(t_n : t_{n-1})\underline{x}(t_{n-1}) \tag{A2}$$

This appendix will explicitly present the transition matrix, $\Phi$, used in this application.

The unforced portion of the solution of equation (A1) is

$$\left.\begin{aligned} x(t) &= e^{-at}x_0 + te^{-at}(\dot{x}_0 + ax_0) \\ \dot{x}(t) &= e^{-at}\dot{x}_0 + te^{-at}(-a\dot{x}_0 - a^2x_0) \end{aligned}\right\} \text{for} \quad N \equiv 0$$

$$\begin{bmatrix} x(t_2) \\ \dot{x}(t_2) \end{bmatrix} = \begin{bmatrix} 1 + a(t_2 - t_1) & (t_2 - t_1) \\ -a^2(t_2 - t_1) & 1 - a(t_2 - t_1) \end{bmatrix} e^{-a(t_2 - t_1)} \begin{bmatrix} x(t_1) \\ \dot{x}(t_1) \end{bmatrix}$$

This is a continuous solution of the homogeneous equation ($N \equiv 0$). It is valid at any time $t_2$ given the initial conditions at $t_1$. Accordingly, the general transition matrix is

$$\begin{bmatrix} x(t_n) \\ \dot{x}(t_n) \end{bmatrix} = \begin{bmatrix} D(1 + aZ) & DZ \\ -Da^2Z & D(1 - aZ) \end{bmatrix} \begin{bmatrix} x(t_{n-1}) \\ \dot{x}(t_{n-1}) \end{bmatrix}$$

where $Z = t_n - t_{n-1}$ and $D = e^{-aZ}$

# APPENDIX B

## SHOWING THAT THE SYMMETRIC THREE BY THREE MATRIX, B, CAN BE EXPRESSED AS $\gamma\gamma^T$ WHERE $\gamma$ IS LOWER TRIANGULAR

The relation to be examined here is

$$\begin{bmatrix} \gamma_{11} & 0 & 0 \\ \gamma_{21} & \gamma_{22} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{31} \\ 0 & \gamma_{22} & \gamma_{32} \\ 0 & 0 & \gamma_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$$

where there are six unknown elements, $\gamma_{ij}$. After matrix multiplication, the necessary six independent equations may be found by equating corresponding matrix elements of the symmetric matricies.

$$\begin{bmatrix} \gamma_{11}^2 & \gamma_{11}\gamma_{21} & \gamma_{11}\gamma_{31} \\ \gamma_{11}\gamma_{21} & \gamma_{21}^2 + \gamma_{22}^2 & \gamma_{21}\gamma_{31} + \gamma_{22}\gamma_{32} \\ \gamma_{11}\gamma_{31} & (\gamma_{21}\gamma_{31} + \gamma_{22}\gamma_{32}) & \gamma_{31}^2 + \gamma_{32}^2 + \gamma_{33}^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$$

and we find that

$$\gamma_{11} = b_{11}{}^{1/2} \qquad \gamma_{22} = [b_{22} - \gamma_{21}^2]^{1/2}$$

$$\gamma_{21} = b_{12}/\gamma_{11} \qquad \gamma_{32} = [b_{23} - \gamma_{21}\gamma_{31}]/\gamma_{22}$$

$$\gamma_{31} = b_{13}/\gamma_{11} \qquad \gamma_{33} = [b_{33} - \gamma_{31}^2 - \gamma_{32}^2]^{1/2}$$

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